Computational Solutions of Feynman Integrals for One-Loop Five-Point Processes

IPP Summer Student at CERN Fellowship Report

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ABSTRACT: We present the computation of the canonical form for the differential equations for the master integrals of three one-loop five-point integrals. The integrals of interest are the one-loop one-mass five-point, the one-loop two non-adjacent mass five-point, and the one-loop two adjacent mass five-point. Integrals such as these are important for the calculation of scattering amplitudes in high energy collider physics. A brief overview of Feynman integrals and their properties is given, along with the concept of IBPs, master integrals, and the development of a pure basis for the master integrals. We also discuss the derivation of the canonical form of the differential equations for the master integrals in the pure basis. In our work, we compute the master integrals associated with each one-loop five-point integral, and discuss how to cast the master integrals into a pure basis. We compute the exact form of the differential equations in this pure basis, this is done using the associated symbol alphabet. Computational work is aided by the Mathematica package `LiteRed`, which computes IBPs, master integrals and derivatives. The possible applications of these integrals, along with plans for the computation of other one-loop five-point integrals are discussed.

Contents

1 Introduction

Calculating scattering amplitudes allows us to predict the results of collider physics, such as the experiments done at CERN. These scattering amplitudes are calculated perturbatively in quantum field theory (QFT), and take the form of Feynman integrals. The leading order term of the perturbation, also known as tree level, is trivial to compute and is not the interest of this paper. The next-to leading order term is a one-loop integral, which can be challenging to compute but is well understood. A significant portion of this project with focus on one-loop processes, as they are an appropriate difficulty to become familiarized with the methods used to solve the Feynman integrals, but well understood enough to ensure our results are correct. The next-to-next-to leading order terms manifests themselves as two-loop integrals, understanding these is an active field of research as they become computationally complex. These higher order terms, one-loop and two-loop, are necessary when predicting the results of high energy collider physics, e.g. see [1], [2] and [3]. The one-loop and two-loop terms take the form of integrals over the phase space of the loop momenta, where the integrand is made up of rational factors of propagators [4].

As an illustrative example, consider the simple case of a massive particle that remains a massive particle. Then the one-loop and two-loop diagrams would appears as in Figure 1. Note, here we only included the planar two-loop example, in general there will be two-loop

(a) One Loop. (b) Planar Two Loops.

Figure 1: Diagrams for a massive particle incoming with momentum p which simply remain that particle.

systems where the loops do not share a two-dimensional plane as in Figure 1. Non-planar two-loop integrals are not included as they are not the focus of this discussion. For an in-depth look at two-loop integrals for planar five-point one-mass processes, see [5]. First, there are a few important things to note when looking at these diagrams.

(a) External momentum is conserved. In this case it is trivial as $p = p$. But, in cases with more than one external momentum, this allows the elimination of one of the external momenta, as it depends on the others. That is, if there is five external momenta, then four of the external momenta are independent due to momentum conservation.

(b) Our choice of loop momenta is arbitrary. For example in the one-loop integral we could have chosen for k to be on the top half of the loop and point counter-clockwise. This would change nothing about the final result.

(c) Momentum is conserved at each vertex. Looking at the one-loop integral in Figure 1, we see on the left side of the diagram where the loop meets the incoming line there is total momentum of $k+p$ entering the vertex, then to preserve momentum conservation there must be total momentum $k + p$ leaving the vertex. This is exactly how we will determine the momenta on various parts of the loop(s) after our initial arbitrary choice.

While the diagrams in Figure 1 do not illustrate any of the underlying mathematical structure of the integrals, one can nonetheless see how the complexity of the system evolves as more loops are added. Now with some familiarity regarding these integrals let us begin considering what we actually want to calculate. What we will be looking at are integrals over the loop momenta. For now we will consider ourselves to be working in four-dimensional spacetime (this will change soon). In that case, for the one-loop and two-loop processes we are interested in calculating something of the form,

$$\int d^4k \Big(\dots \Big) \qquad \text{One Loop,} \tag{1.1}$$

$$\int \int d^4l \, d^4k \Big(\dots \Big) \qquad \text{Two Loops.} \tag{1.2}$$

The terms that we will see in the integrands relate to the momenta of each internal leg of the loop. For the scalar integrals we are interested in, the integrand will be the product of the momenta of each internal leg raised to the power of −2. That is,

$$\int d^4k \bigg(\frac{1}{k^2(k+p)^2} \bigg) \qquad \text{One Loop,} \tag{1.3}$$

$$\int\int d^4l\, d^4k \bigg(\frac{1}{l^2k^2(k+p-l)^2} \bigg) \qquad \text{Two Loops.} \tag{1.4}$$

Note that if an internal leg were to have a mass, then the associated term in the integral would change. Suppose that the upper internal leg of the one-loop integral in Figure 1 was massive, with mass m, then the associated term in the integrand would be of the form

$$\frac{1}{(k+p)^2 - m^2}. \tag{1.5}$$

The integrals in Equation 1.3 and Equation 1.4 are exactly the form of integrals we are interested in solving. To familiarize ourselves with the methods used to solve these integrals we will proceed with the one-loop integral from Figure 1, now simply referred to as the massive bubble.

1.1 Properties of the Integrals

First, to solve these integrals we consider a more general case. We begin by generalizing the dimensionality ($4 \to D$), this is known as dimensional regularization and will solve the following issue: the integrals diverge in four dimensions [4]. We will later make the substitution $D = 4 - 2\epsilon$ and expand into a Laurent series in ϵ. Doing this allows the divergence to be removed. Secondly, we consider the generalized form of the scalar integral as

$$\text{Bub}(\nu_1, \nu_2) = \int \frac{d^Dk}{[k^2]^{\nu_1}[(k+p)^2]^{\nu_2}}. \tag{1.6}$$

The generalization of the powers which the propagators are raised to, will allow us to use integration by parts relations (IBPs) [6]. These relations relate integrals in this family of bubble integrals to each other, for example, letting us write Bub(2, 3) as a linear combination of other bubble integrals. This quickly leads to the idea of master integrals, that any bubble integral can be written as a linear combination of a specific set of bubble integrals, which span the set of all bubble integrals.

Another important property of the Feynman integrals is their Lorentz invariance, therefore they are only functions of Lorentz invariants. In a system with more than just the one momenta, that would mean they are functions of p_ip_j. In our case, the massive bubble only has the invariant p^2. We will use the notation $s_{ij} = (p_i + p_j)^2$ going forward, as our integrals will depend on these invariants, which are known as Mandelstam variables, and will use the notation $\vec{s}$ to refer to a vector consisting of all the invariants. We now state two important results from dimensional regularization.

(a) All scaleless integrals are zero.

(b) All total derivatives are zero.

Thus, IBPs which will define recursion relationships within the family of integrals is given, for a general one loop system, as [6]

$$\int d^D k \frac{\partial}{\partial k^\mu}\left(v^\mu \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2\ldots}\right) = 0. \tag{1.7}$$

Where $v = k, p_1, p_2, \ldots$. This expression will allow recursion relations to be derived which lead to the master integrals of the process. For more complex systems, the relations become technically demanding, and are best dealt with computationally. For the massive bubble example there is two choices: $v = k$ or $v = p$. We will state both results, but note that differentiation brings terms into the numerator, this actually leads to the shifting of powers of ν_m $(m = 1, 2)$ and is exactly how the IBPs are found. Applying Equation 1.7 to the massive bubbles gives

$$(D - 2\nu_1 - \nu_2)\mathrm{Bub}(\nu_1, \nu_2) - \nu_2 \mathrm{Bub}(\nu_1 - 1, \nu_2 + 1) + \nu_2 p^2 \mathrm{Bub}(\nu_1, \nu_2 + 1) = 0, \tag{1.8}$$

$$\begin{aligned}(\nu_1 - \nu_2)\mathrm{Bub}(\nu_1, \nu_2) + \nu_1 p^2 + \nu_1 p^2 \mathrm{Bub}(\nu_1 + 1, \nu_2) - \nu_2 p^2 \mathrm{Bub}(\nu_1, \nu_2 + 1) \\ - \nu_1 \mathrm{Bub}(\nu_1 + 1, \nu_2 - 1) + \nu_2 \mathrm{Bub}(\nu_1 - 1, \nu_2 + 1) = 0.\end{aligned} \tag{1.9}$$

Where the first equation comes from setting $v = k$ and the second from setting $v = p$. These two equations can be manipulated to give the recursion relations,

$$\mathrm{Bub}(\nu_1, \nu_2) = \frac{\nu_1 + \nu_2 - 1 - D}{p^2(\nu_2 - 1)}\mathrm{Bub}(\nu_1, \nu_2 - 1) + \frac{1}{p^2}\mathrm{Bub}(\nu_1 - 1, \nu_2), \tag{1.10}$$

$$\mathrm{Bub}(\nu_1, \nu_2) = \frac{\nu_1 + \nu_2 - 1 - D}{p^2(\nu_1 - 1)}\mathrm{Bub}(\nu_1 - 1, \nu_2) + \frac{1}{p^2}\mathrm{Bub}(\nu_1, \nu_2 - 1). \tag{1.11}$$

Notice here that the first relation cannot be used for $\nu_2 = 1$ and the second cannot be used for $\nu_1 = 1$. From these recursion relationships, it becomes clear how we can write any integral in this family in terms of other integrals.

1.2 Differential Equations of the Master Integrals

To solve the integrals, the same method as in [5] will be used, which follows from [7], [8], [9], [10], [11], and [12]. The goal will be to find the master integrals, and then derive differential equations for the master integrals. Then a pure basis is found for the master integrals. This differential equations for the pure basis takes the canonical form [13]

$$\mathrm{d}\vec{I} = M\vec{I}, \qquad M = \epsilon \sum_\alpha M_\alpha \,\mathrm{d}\log(W_\alpha). \tag{1.12}$$

Where $\vec{I}$ is the column vector of the master integrals in the pure basis, and the matrices M_α are made up of rational numbers. Also the terms W_α are functions of the kinematics of the process, and will be hence forth be referred to as letters, and the set of all letters known as the alphabet. Notice here that the pure basis enables $\mathrm{d}\vec{I}$ to be written as a linear combination of d-log-forms. Then the goal will be to find this pure basis, and then find the exact form of all M_α. In our work, we will be proceeding with the letters already known, so this report will not include deriving the letters of the process. But, the dimension of the alphabet will be calculated as a check on the procedure. We form this pure basis by using prior experience, that is, we know how to correctly write various integrals in such a way that we will arrive at the canonical form. Deriving the correct form for each of these integrals is outside the scope of this report. For a discussion on how to form the pure basis see [5].

Suppose now that we have a pure basis of master integrals, and the alphabet is known. Then our objective is to solve all the M_α. To do so, we use numerical evaluation of random points in phase space, while also fixing the value of ϵ. With enough of these evaluations, the M_α will become fully constrained, this method follows from [5] and [7]. We begin by forming a random direction differential equation. Let $\vec{c}$ be some vector in the phase space of the process, then we introduce the differential equation

$$\vec{c} \cdot \frac{\partial \vec{I}}{\partial \vec{s}} = C(\epsilon, \vec{s})\vec{I}. \tag{1.13}$$

This differential equation allows the use of numerical evaluations, unlike the total derivative in Equation 1.12. In the pure basis, it follows that $C(\epsilon, \vec{s})$ takes the particular form

$$C(\epsilon, \vec{s}) = \epsilon \sum_\alpha M_\alpha \; \vec{c} \cdot \frac{\partial}{\partial \vec{s}} \log(W_\alpha). \tag{1.14}$$

We will then evaluate the left-hand side of Equation 1.13 over at least as many points as there are letters. Doing so allows $C(\epsilon, \vec{s})$ to be evaluated at just as many points, so with the letters known, it becomes possible to solve all of the M_α using Equation 1.14. Giving a fully analytic form of the differential equations for each of the master integrals.

In the numerical evaluation, the dimension of the alphabet will also be calculated. To do so, we evaluate $C(\epsilon, \vec{s})$ at m random points, $\vec{s}_1, \ldots \vec{s}_m$, in phase space. Then flatten this matrix into a column vector. We then construct a new matrix formed from these column vectors. Evaluating the rank of this new matrix will give the minimum of m and the dimension of the alphabet [5]. Hence we repeat this process while increasing the value of m, eventually the rank of the matrix will plateau and thus the dimension of the alphabet will be found. Doing this will ensure the alphabet we are using is of the correct dimension.

2 Methods

This report will focus on three different one-loop five-point integrals. The first with one mass, the second with two non-adjacent masses, and the third with two adjacent masses. In this section we will discuss the transformations needed to construct the pure basis for the different integrals. The master integrals for each of these five-point integrals will contain various bubbles, triangles, boxes and pentagons. This process will rely heavily on

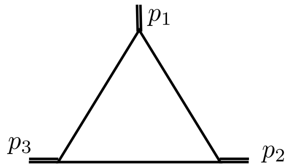


Figure 2: A triangle with three massive legs, that is, p_1^2, p_2^2 and p_3^2 are all nonzero.

the Mathematica package `LiteRed`, which will preform the IBP reduction, find the master integrals, and handle differentiation with respect to the Mandelstam variables [14] [15]. While the rest of the numerical evaluation will be performed using Mathematica.

2.1 Constructing the Pure Basis

To cast the master integrals into the pure basis, the integrals will be multiplied by a term which is a function of ϵ and the Mandelstam variables. For simplicity, we will not be including arrows on the diagrams of the various processes. As a convention we will consider all external momenta to be pointed inwards, the direction of the external momenta does not affect the final differential equations and thus is not our concern. Also, we will denote a massive external leg pictorially by a double line for the external leg, as opposed to a single line for a massless external leg. The simplest of the integrals that will appear in the list of master integrals are the two-point integrals (bubbles), which is cast into the pure basis as

$$\text{Bub} \to \epsilon(1-2\epsilon)\text{Bub}, \tag{2.1}$$

which works for both massless and massive bubbles, as long as the internal legs are massless. Note the right-hand side of Equation 2.1 is how we write the two-dimensional bubble in four-dimensions, this is because the bubble integral is already pure in two-dimensions.

Now let us consider the three-point integrals (triangles). For this report, we need only worry about a three-mass triangle. The other triangles, one-mass and two-masses, do not appear as master integrals in the one-loop five-point integrals of interest. For the three-mass triangle see Figure 2. To cast such a three-mass triangle into the pure basis we multiple the three-mass triangle integral, Tri_3, as

$$\text{Tri}_3 \to \epsilon^2 \sqrt{-\det \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2 \cdot p_2 \end{pmatrix}}\, \text{Tri}_3. \tag{2.2}$$

Next are the four-point integrals (boxes), of which there is four cases we will need to consider. There is the one-mass box, two different two-mass boxes, and lastly a three-mass box that will appear in the list of master integrals. First, consider the one-mass box from Figure 3. To cast the one-mass box into the pure basis, we multiple the one-mass box integral, Box_1, as

$$\text{Box}_1 \to \epsilon^2 s_{23} s_{34}\, \text{Box}_1. \tag{2.3}$$

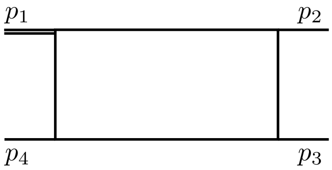


Figure 3: A box with one massive leg, that is, p_1^2 is nonzero.

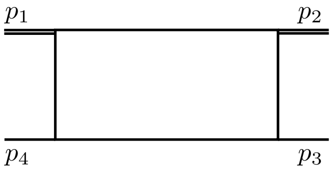


Figure 4: A box with two massive legs, that is, p_1^2 and p_2^2 are nonzero.

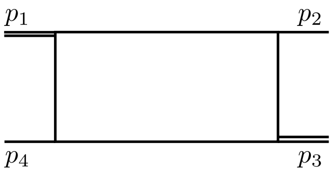


Figure 5: A box with two massive legs, that is, p_1^2 and p_3^2 are nonzero.

Now consider the two-mass box with the two masses adjacent to each other, this is also known as the hard box, see Figure 4. To cast the hard box into the pure basis, we multiple the hard box integral, $\text{Box}_{2-\text{hard}}$, as

$$\text{Box}_{2-\text{hard}} \to \epsilon^2 s_{12} s_{23} \, \text{Box}_{2-\text{hard}}. \tag{2.4}$$

The other two-mass box case is when the massive external legs are opposite of each other, this is also known as the easy box, see Figure 5. To cast the easy box into the pure basis, we multiple the easy box integral, $\text{Box}_{2-\text{easy}}$, as

$$\text{Box}_{2-\text{easy}} \to \epsilon^2 \left(s_{12} s_{23} - p_1^2 p_3^2 \right) \text{Box}_{2-\text{easy}}. \tag{2.5}$$

The last box we must consider is the three-mass box. There is only one unique permutation of the external legs, just as with the one-mass box. For the three-mass box, see Figure 6. To cast the three-mass box integral into the pure basis we multiple the integral, Box_3, as

$$\text{Box}_3 \to \epsilon^2 \left(s_{12} s_{23} - p_1^2 p_3^2 \right) \text{Box}_3. \tag{2.6}$$

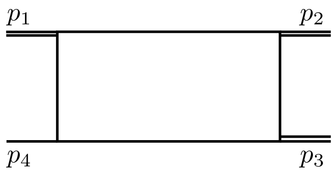
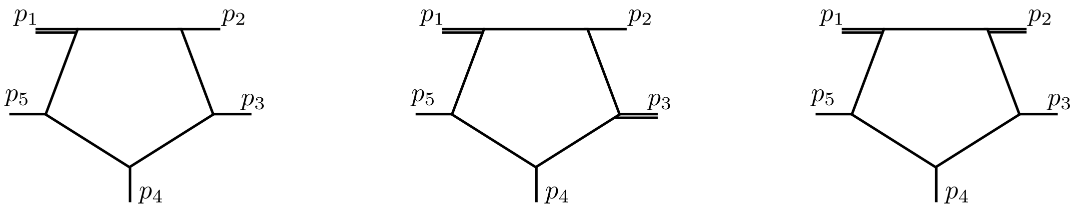


Figure 6: A box with three massive legs, that is, p_1^2, p_2^2, and p_3^2 are nonzero.



(a) One-mass five-point. (b) Two non-adjacent mass five-point. (c) Two adjacent mass five-point.

Figure 7: The different one-loop five-point integrals we will see appear in the list of master integrals.

The last type of integral we must consider is that of the five-point integrals. The correct normalization for these integrals will be the same regardless of the masses. For examples of pentagons we will encounter as master integrals see Figure 7. We will intermediately define the following matrices to simplify the notation,

$$A = \begin{pmatrix} p_1 \cdot p_1 & \dots & p_1 \cdot p_4 \\ \vdots & \ddots & \vdots \\ p_4 \cdot p_1 & \dots & p_4 \cdot p_4 \end{pmatrix}, \tag{2.7}$$

$$B = \begin{pmatrix} k \cdot k & \dots & k \cdot p_4 \\ \vdots & \ddots & \vdots \\ p_4 \cdot k & \dots & p_4 \cdot p_4 \end{pmatrix}. \tag{2.8}$$

Where k is the loop momenta of the pentagon. Note here that in the B matrix, scalar products such as $k \cdot p_i$ appear. To handle these, we rewrite these scalar products in terms of the propagators. These relationships can be worked out by hand, or simply given by `LiteRed`. These relationships will be different depending on which external and internal legs of the pentagon are massive or massless. The correct factor to cast a pentagon integral, Penta, into the pure basis is [16]

$$\text{Penta} \to \epsilon^2 \left(\frac{\det B}{\sqrt{\det A}} \right) \text{Penta}. \tag{2.9}$$

Note here that because $\det B$ contains factors of the propagators, there will be a linear combination of various integrals when the right-hand side of Equation 2.9 is expanded. To handle this, we use LiteRed to reduce the resulting equation to the master integrals using IBP reduction. The right-hand side of Equation 2.9 is how we write the six-dimensional pentagon in four-dimensions, this is because the pentagon integral is already pure in six-dimensions.

2.2 Obtaining the Analytical Form of the Differential Equations

Using LiteRed we are able to compute the derivatives of the master integrals with respect to each of the Mandelstam variables. But these derivatives are not in the pure basis. To obtain the matrix M in Equation 1.12, in the pure basis, we use the result from the original master integral basis along with the transformation T, which we use to go from the original basis of master integrals to the pure basis. Let A_{ij} be the derivative of the original basis with respect to s_{ij}, and B_{ij} defined similarly but in the pure basis. Then we obtain the various B_{ij} as

$$B_{ij} = \left(\frac{\partial T}{s_{ij}}\right)T^{-1} + TA_{ij}T^{-1}. \tag{2.10}$$

While naively one might think at this step we are done since we have M, in general, we cannot simply integrate the right-hand side of Equation 1.12. This is exactly why we work in the pure basis, since the differential equations take on the more manageable canonical form. So we must then next derive the M_α and W_α. We derive the letters, W_α from the above B_{ij}, but do not cover this process in the report, see [5]. Once the letters are obtained our work is quickly completed as described in subsection 1.2. We follow the same procedure as in [5] and [7], where we first compute the matrix of random direction d logs,

$$\mathcal{W}_{\alpha k} = \vec{c} \cdot \left[\frac{\partial}{\partial \vec{s}} \log(W_\alpha)\right]_{\vec{s}=\vec{s}^{(k)}}. \tag{2.11}$$

We are then able to compute each M_α using the inverse of this matrix. The inverse $\mathcal{W}_{\alpha k}$ exists since the d logs are independent. The independence of the d logs is verified numerically as an intermediate step. Then we can calculate the M_α exactly as,

$$M_\alpha = \sum_k \frac{1}{\epsilon} \mathcal{W}^{-1}_{\alpha k} C(\epsilon, \vec{s}^{(k)}). \tag{2.12}$$

The results of these matrices will be exactly rational numbers. Hence we will have found the canonical form of the differential equations of the master integrals.

3 Results

As stated before the focus of this report will be the three different one-loop five-point integrals in Figure 7. In this section we will state the results of our work, this will consist of the master integrals of each five-point integral, the dimension of the alphabet, the unique rational numbers found in the M_α, and with the number of non-zero entries found in the M_α. It would not be enlightening to present the M_α matrices in their entirety as there

	Unique Rational Numbers	Number of Non-Zero Entries
One-Mass	$0, \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$	129
Two Non-Adjacent Mass	$0, \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$	130
Two Adjacent Mass	$0, \pm 1, \pm 2, 4, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}, \pm\frac{1}{16}$	161

Table 1: The unique rational numbers, and number of non-zero entries found in the M_α matrices for the one-mass five-point integral, the two non-adjacent mass five-point integral and the two adjacent mass five-point integral.

is 30 13×13 matrices, 39 15×15 matrices, and 43 16×16 matrices, while the non-zero entries are sparse with few unique entries.

For the one-loop five-point integral with a single mass, there was 13 master integrals found. They consist of six bubbles (one for each of the Mandelstam variables), one three-mass triangle, two one-mass boxes, three two-mass boxes, and the one-mass pentagon itself. To see diagrams of these master integrals please refer to Appendix A. The dimension of the alphabet for this integral was found to be 30, so there was 30 letters needed along with 30 M_α's to fully describe the canonical form of the differential equation in Equation 1.12. To see the unique rational numbers and the number of non-zero entries in the M_α found for this five-point integral, see Table 1.

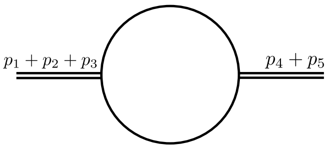
The one-loop five-point integral with two non-adjacent masses had 15 master integrals found. They were seven bubbles (one for each of the Mandelstam variables), two three-mass triangle, four two-mass boxes, one three-mass box, and the two non-adjacent mass pentagon itself. The diagrams for these master integrals are given in Appendix B. The dimension of the alphabet for this integral was found to be 39, hence with all 39 letters and M_α matrices we are able to write the entirety of the canonical form of the differential equation in Equation 1.12. The results found for these M_α matrices are given in Table 1.

The one-loop five-point integral with two adjacent masses was found to have 16 master integrals. They consist of seven bubbles (one for each of the Mandelstam variables), three three-mass triangle, three two-mass boxes, two three-mass box, and the two adjacent mass pentagon itself. See Appendix C for the diagrams of these master integrals. The length of the alphabet was computed to be 43, hence with 43 letters and 43 M_α's we have the canonical form of the differention equation from Equation 1.12. Refer to Table 1 for the unique rational numbers found in the M_α along with the number of non-zero entries.

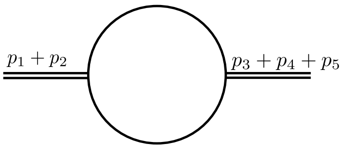
4 Conclusions

In this report we briefly covered the basics of Feynman integrals, and their importance when calculating scattering amplitudes in experiments, such as the ones done at CERN. We saw the use of dimensional regularization, the implementation of IBPs, and the idea of master integrals. We then built off the work of [5], to develop a procedure that would generate the canonical form of the differential equations for the pure basis of master integrals. This procedure required the development of a pure basis for the master integrals, the derivation of the associated letters, and finally the generation of the M_α matrices found in Equation 1.12. In detail, we discussed the factors one would use to cast various master integrals into the pure basis. In particular, we were able to do this for a one-loop five-point one-mass integral, along with the two unique cases of the one-loop five-point two-mass integral. For these cases, we stated the number of master integrals found, the length of the alphabet, the unique rational numbers present in the M_α matrices, and the number of non-zero terms in the M_α matrices. With the data we were able to produce, one could follow the work of [5] to numerically evaluate the master integrals. This procedure could easily be applied to other processes, in particular one-loop five-point processes with an internal mass present, in fact, these are the processes we hope to apply the procedure to next. At the time of writing we are seeing promising results for one-loop five-point processes with an internal mass. Integrals like the ones we present in this report, along with the ones we have began work on, could be applied to experiments. In particular, the one-loop two-mass five-point integrals we worked with could represent a process for the production of two heavy bosons along with a jet. While the one-loop two-mass five-point integrals with an internal mass we have begun work on represent a $t\,\bar{t}$ plus jet production process. Processes like these could be applied to the experiments done at CERN.

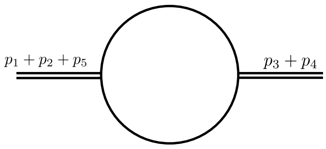
A Master Integrals of the One-Loop Five-Point One-Mass Process



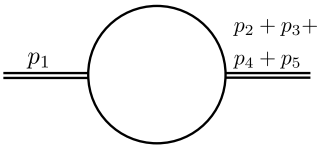
(a) A massive bubble.



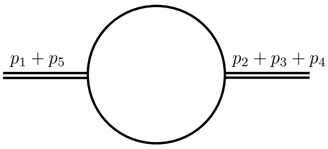
(b) A massive bubble.



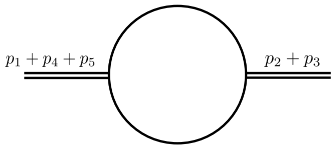
(c) A massive bubble.



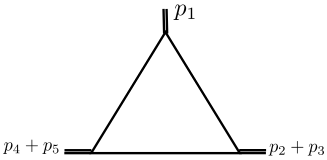
(d) A massive bubble.



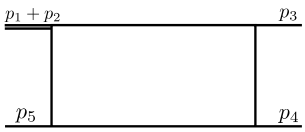
(e) A massive bubble.



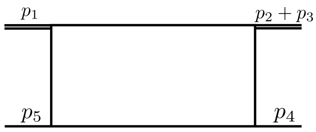
(f) A massive bubble.



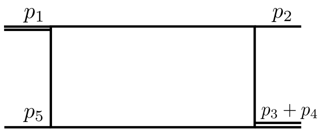
(g) A three-mass triangle.



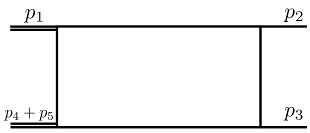
(h) A one-mass box.



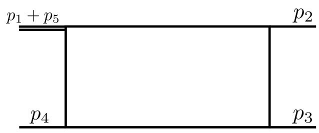
(i) A two-mass box.



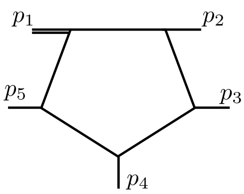
(j) A two-mass box.



(k) A two-mass box.



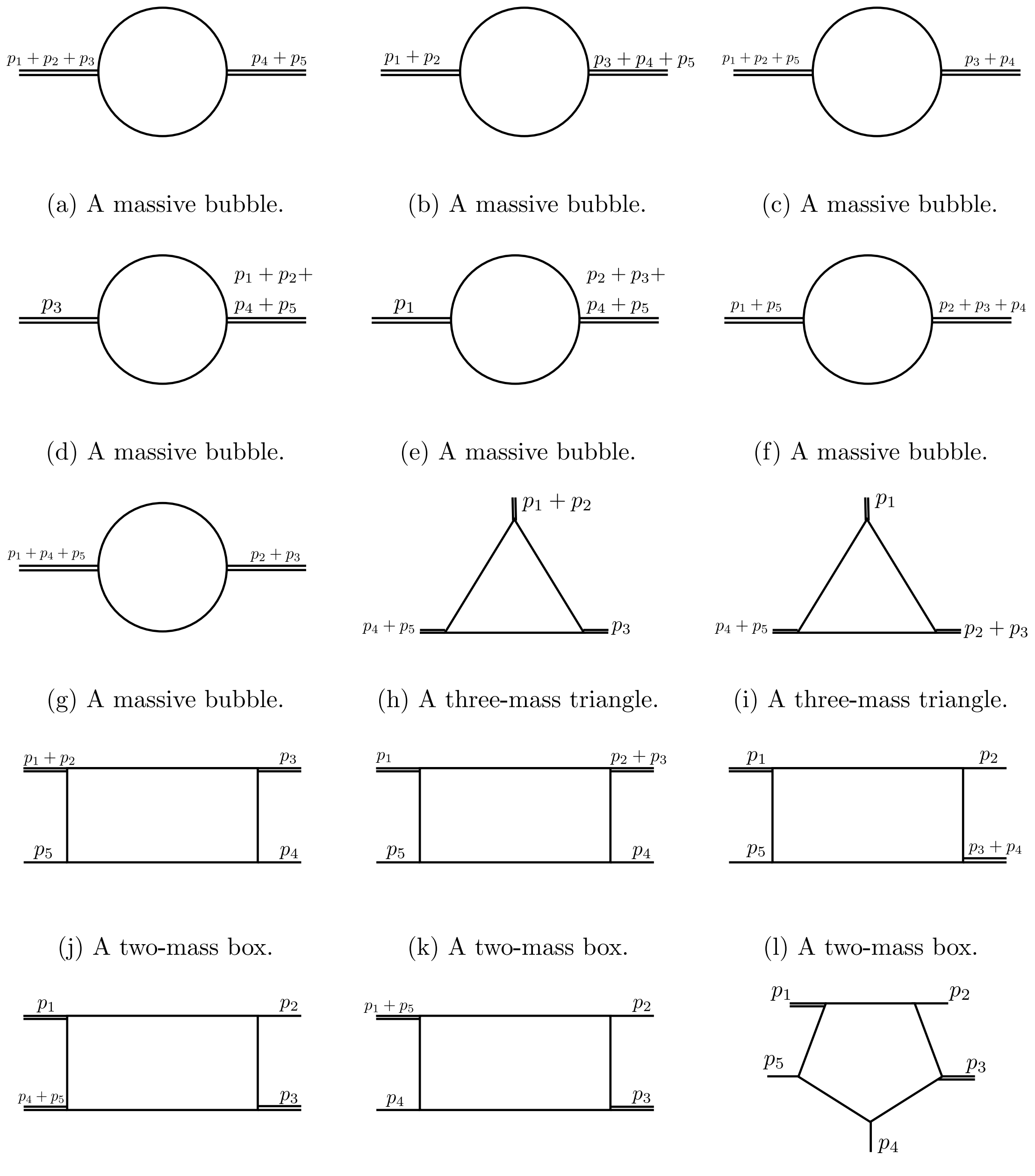
(l) A one-mass box.



(m) The one-mass five-point.

Figure 8: The master integrals found for the one-loop five-point one-mass process.

B Master Integrals of the Two-Loop Five-Point Two Non-Adjacent Mass Process



(a) A massive bubble. (b) A massive bubble. (c) A massive bubble.

(d) A massive bubble. (e) A massive bubble. (f) A massive bubble.

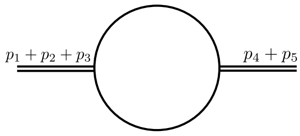
(g) A massive bubble. (h) A three-mass triangle. (i) A three-mass triangle.

(j) A two-mass box. (k) A two-mass box. (l) A two-mass box.

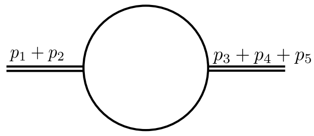
(m) A three-mass box. (n) A two-mass box. (o) The two-mass five-point.

Figure 9: The master integrals found for the one-loop five-point two non-adjacent mass process.

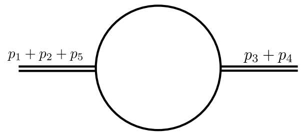
C Master Integrals of the Two-Loop Five-Point Two Adjacent Mass Process



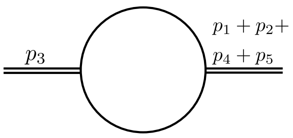
(a) A massive bubble.



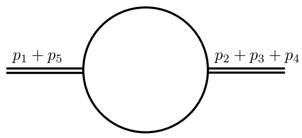
(b) A massive bubble.



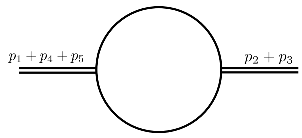
(c) A massive bubble.



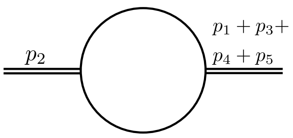
(d) A massive bubble.



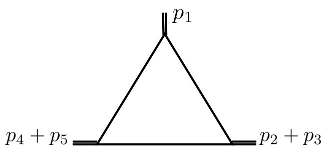
(e) A massive bubble.



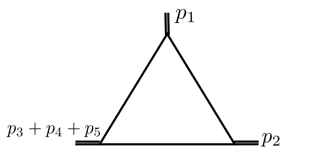
(f) A massive bubble.



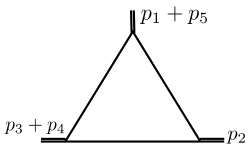
(g) A massive bubble.



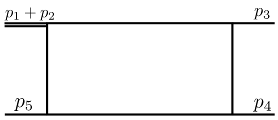
(h) A three-mass triangle.



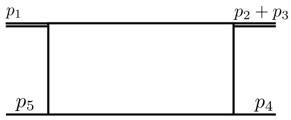
(i) A three-mass triangle.



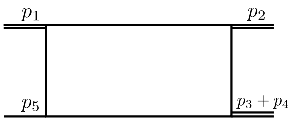
(j) A three-mass triangle.



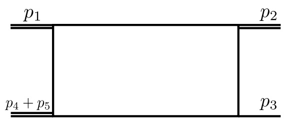
(k) A one-mass box.



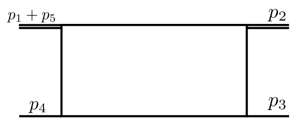
(l) A two-mass box.



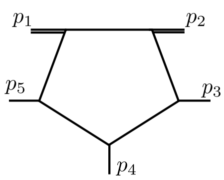
(m) A three-mass box.



(n) A three-mass box.



(o) A two-mass box.



(p) The two-mass five-point.

Figure 10: The master integrals found for the one-loop five-point two adjacent mass process.

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