# Constraining Scattering Amplitudes in the Asymptotic Regge Limit

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## ABSTRACT

This report briefly introduces analytic constraints on scattering amplitude and summarizes research that was conducted during July and August of 2018 in the Theoretical Physics Group at CERN. The work was inspired by new positivity properties of Gegenbauer polynomials developed by N. Arkani-Hamed, Y. Huang, T. Huang, and S. Shao. Their methods and results were reproduced and are summarized; while their application was to effective field theories, the research described in this report investigates similar positivity constraints in higher energy regimes. In particular, previous constraints were reproduced using these new methods, though no additional constraints were obtained.

## BACKGROUND

The study of scattering amplitudes is a rapidly growing field due to the strength of its techniques and its connection to many areas of high energy physics research. Scattering amplitudes are formulae that describe how elementary particles interact. In practice, we concern ourselves with relativistic theories, so quantum field theories (QFTs) are generally used to calculate these amplitudes. For those familiar with QFT, scattering amplitudes are the 'things' you compute using Feynman diagrams and Feynman rules. Scattering amplitudes are so important because they relate a theory to experiment. The theory predicts the amplitude and the experiment measures the cross-section. These two quantities are related by $d\sigma/d\Omega \propto |A|^2$; integrating the modulus squared of the amplitude over all solid angles gives the total cross-section of the interaction [1].


Figure 1: A typical scattering of elementary particles.

In the theory of amplitudes, we study an object called the S-matrix, which describes the interaction of particles and contains the information given by the scattering amplitudes. Its matrix elements are defined by $S_{i,j} := \langle i|S|j\rangle$, where $|j\rangle$ is an initial state and $\langle i|$ is a final state. In practice, the input and output states are of a continuous variable and hence one thinks of the S-matrix as a space of functions, the scattering amplitudes, rather than a discrete matrix.

For experiments at the Large Hadron Collider, the main quantum field theory that governs particle interactions is Quantum Chromodynamics (QCD). This theory describes the strong interaction governing the dynamics of quarks and gluons, the ingredients of hadrons; it is an extremely complex theory. For instance, the theory of nuclear physics can be seen as a special case of QCD. Because of this complexity, there is no hope of calculating the S-matrix exactly. The problem solving strategy, to decipher the predictions of the theory, is to generalize the number of colours, N, and consider the limit in which N is large. One then hopes that N=3 is large enough to connect the large N theory to the N=3 reality (in QCD there are three colours) [2]. In this large N expansion, the tree-level diagrams dominate and the higher-level terms are of order 1/N. Thus at large N, one only needs to consider the tree-level processes. In this case, the analytic extension of the amplitude becomes a meromorphic function of $s$ and $t$, the Mandelstam variables, with poles at real masses corresponding to a virtual exhange particle going on-shell. The variable $s$ encodes the energy and $t$ encodes the momentum transfer. Then, since the amplitude has specified values at infinity, the residues at the poles completely fix the amplitude.

There are two commonly applied constraints on scattering amplitudes that allow for general statements about the amplitude's mathematical properties. The first is unitarity of the S-matrix,

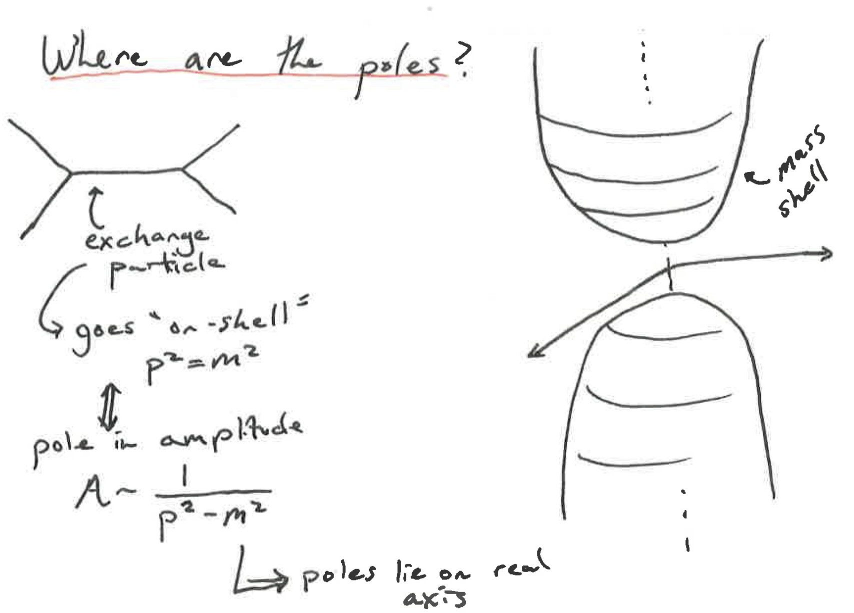


Figure 2: The poles of the amplitude correspond to the exchange of a particle of mass $m$.

namely that $S^*S = SS^* = \mathbb{1}$. This is just a statement imposing conservation of probabilities: $(S\psi)^*(S\psi) = \psi^*S^*S\psi = \psi^*\psi$. This manifests as the fact that the residues of the amplitude are sums of Legendre polynomials with positive coefficients,

$$\lim_{t\to m_i^2} A(s,t) = \frac{1}{t-m_i^2}\sum_{j=0}^{L_i} f_{i,j}^2 P_j(1+\frac{2s}{m_i^2}). \tag{1}$$

The second constraint is that of crossing symmetry: $A(s,t) = A(t,s)$, which states that a permutation of particle labelling results in the same physical process.

The reported research was concerned with a high-energy limit in which $s$ and $t$ are both large and positive, providing information about heavy spinning resonances in higher spin theories [3]. In this regime, the amplitude is written as $A(t,\beta) \sim e^{tf(\beta)}$ for some function $f(\beta)$, where $\beta = s/t$. We notice that setting $f(s/t) = (s+t)\log(s+t) - s\log s - t\log t$ gives the limit of the Veneziano amplitude in the large $s$ and $t$ limit [3]. The goal of the project was to put constraints on $f(\beta)$ which then constrains the amplitude to some required form.

## RECENT DEVELOPMENTS

In early 2018, Nima Arkani-Hamed and Yu-tin Huang introduced a positivity constraint on scattering amplitudes which was a result of the total positivity (all minors being positive) of the following matrix:

$$\begin{bmatrix} P_0(1+2\beta) & P_1(1+2\beta) & P_2(1+2\beta) & \dots \\ \partial_\beta P_0(1+2\beta) & \partial_\beta P_1(1+2\beta) & \partial_\beta P_2(1+2\beta) & \\ \partial^2_\beta P_0(1+2\beta) & \partial^2_\beta P_1(1+2\beta) & \partial^2_\beta P_2(1+2\beta) & \\ \vdots & & & \ddots \end{bmatrix}, \tag{2}$$

where $P_i$'s are Gegenbauer polynomials and $\beta$ is related to the scattering angle $\theta$ by $\cos\theta = 1+2\beta$ [4][5]. The following section documents the relations between this new positivity and high-energy positivity constraints introduced in [3]. The goal was to use this new method as inspiration for possible new constraints enforced by matrix positivity properties.

## ASYMPTOTIC REGGE REGIME

In the Arkani-Hamed and Huang matrix, the functions were evaluated at $\theta = 0$. The first computation done ensured that the total positivity held for the imaginary scattering angles appearing in the Regge limit, which corresponded to replacing $\cos\theta$ with $\cosh\theta$ in the arguments of the Gegenbauer polynomials. Indeed, total positivity held when derivatives were taken with respect to both $\theta$ and $\beta$ and evaluated at imaginary angles. Realize that the equivalence of the results between the two derivative variables is expected since $\partial_\theta = \sqrt{\beta(1+\beta)}\partial_\beta$. The matrix considered is shown below:

$$\begin{bmatrix} P_0(\cosh\theta) & P_1(\cosh\theta) & P_2(\cosh\theta) & \dots \\ \partial_\theta P_0(\cosh\theta) & \partial_\theta P_1(\cosh\theta) & \partial_\theta P_2(\cosh\theta) & \\ \partial^2_\theta P_0(\cosh\theta) & \partial^2_\theta P_1(\cosh\theta) & \partial^2_\theta P_2(\cosh\theta) & \\ \vdots & & & \ddots \end{bmatrix} \tag{3}$$

As discussed earlier, at large $s$ and $t$ the scattering amplitude can be written as $A = e^{tf(\beta)}$ for some function $f(\beta)$. By making the identification $t = j$ (for a linear Regge trajectory) and realizing that in three dimensions the $n^{\text{th}}$ Legendre polynomial can be written as $\cosh(n\theta)$, from the discussion above it is natural to consider the following matrix:

$$\begin{bmatrix} e^{tf(\beta)} & \cosh((1+j)\theta) & \cosh((2+j)\theta) \\ \partial_\theta e^{tf(\beta)} & \partial_\theta \cosh((1+j)\theta) & \partial_\theta \cosh((2+j)\theta) \\ \partial^2_\theta e^{tf(\beta)} & \partial^2_\theta \cosh((1+j)\theta) & \partial^2_\theta \cosh((2+j)\theta) \end{bmatrix} \tag{4}$$

Now by the positivity of matrix elements, we see that since (2,1) entry of the matrix is $e^{tf(\beta)}\sqrt{\beta(1+\beta)}tf'(\beta)$ and since $\beta, t \geq 0$ we find that the derivative $f'(\beta)$ is positive. This corresponds to equation 4.3 in [3]. Additionally, by requiring the minors be positive, in the large $t$ limit, we find that for consistency we must have $\sqrt{\beta(1+\beta)}f'(\beta) \leq 1$. This second inequality appears as a comment after equation 4.9 in [3]. Moreover, one finds that if this constraint is satisfied then the matrix is necessarily totally positive. Considering higher dimensions of this matrix and their minors did not provide any constraints involving terms with higher than a first derivative.

To find constraints involving higher derivatives of $f$, the following matrix was considered:

$$\begin{bmatrix} A & \partial_\beta^{2n} A \\ \partial_\beta^{2n} A & \partial_\beta^{4n} A \end{bmatrix} \tag{5}$$

for some natural number $n$. One sees that this matrix corresponds to the Hankel matrix of moments of A and hence the determinant is known to be positive. With $A = e^{tf(\beta)}$, this statement forces $f''(\beta) > 0$, which, when combined with the discussion above, reproduces equation 4.10 in [3]. Repeating the procedure with higher matrix dimensions, different choices of $n$ or $\theta$-derivatives does not provide additional constraints; second or first derivative terms always dominate.

We next study

$$\begin{bmatrix} A & \partial_\theta A & \partial_\theta^2 A \\ \partial_\theta A & \partial_\theta^2 A & \partial_\theta^3 A \\ \partial_\theta^2 A & \partial_\theta^3 A & \partial_\theta^4 A \end{bmatrix}. \tag{6}$$

One can directly check that if $A$ is some sum of Legendre polynomials, say $A = c_1 \cosh(n\theta) + c_2 \cosh(m\theta)$, the determinant is guaranteed to be positive. Then by taking $A = e^{tf(\beta)}$ we find that at large t, the determinant is dominated by $f'$ and $f''$. Thus, no interesting constraints are found since we know already that both terms are positive.

We noticed a slight variation to matrix (4) in which the number of derivatives at each element is such that the sum of the number of derivatives taken in the (1,1) and (2,2) elements is equal to the analogous sum with the (1,2) and (2,1) elements. Then, assuming that the number of derivatives increases in each row and column, the first derivative term (which has the highest power of $t$) will cancel in the determinant and will therefore not dominate the expression. A similar pattern was sought for the 3x3 case; by requiring the minors to follow the 2x2 pattern, so the first derivatives cancel, is there a way to cancel the second derivative terms? In both 3x3 and 4x4 cases, it was determined not to be possible.

A more general problem is to find differential operators to act on the amplitude and cancel the first derivative terms. By considering higher derivatives of the amplitude and systematically accounting for terms with only $f'$ and $f''$ by subtracting products of lower amplitude derivatives, the second derivatives can be removed but the remaining dominating terms were single $f^{(n)}$ terms (rather than relations between derivatives) which are certainly not non-negative by direct computation. Note that calculating minors of the matrices above are a special and specific cases of this procedure.

Next, crossing symmetry ($\beta \to \frac{1}{\beta}$) was applied to $A = e^{tf(\beta)}$ in the already-considered matrices. From matrix (4), with $\theta$ derivatives taken instead of $\beta$ derivatives (whose determinants were dominated by $f''$ and a positive prefactor), we find the additional constraint on $f$

$$(2+\beta)(f(\beta)-\beta f'(\beta))+2\beta^2(1+\beta)f''(\beta) \geq 0. \tag{7}$$

Note that this constraint can also be found by considering the leading term $t$ in the determinant of (5) under crossing or taking its the first 2x2 minor. If $\theta$ derivatives are replaced by $\beta$ derivatives, $f''$ terms dominate again. We note here that the Veneziano amplitude satisfies this constraint, however the equation does not put any additional constraints on the moments in the multipole expansion of $f(\beta)$.

The main problems with results from the determinants investigations was the lack of higher derivatives appearing in the leading terms at large $t$. Several additional matrices were tested to try to eradicate this issue. These include weighting certain columns or rows with $t$, replacing $\cosh((j+1)\theta)$ by $e^{(t+1)f(\beta)}$ in matrix (3), and considering a similar matrix to (3) but with the first two columns populated by the first two Legendre polynomials and the last by the crossed amplitude.

## CONCLUSIONS

The positivity of the Hankel matrix provides new constraints on the expansion of scattering amplitudes in effective field theories. The properties of this matrix were extended to study the analytic properties of amplitudes in the asymptotic Regge limit of higher spin theories. The interesting result was that known constraints were able to be reproduced using matrix methods. This adds additional detail to why the properties shown earlier must be true. It was shown that using this method in the form decribed above does not yield additional constraints on the moment expansion of the amplitude. Furthur research could focus on connecting the matrix positivity to bounds on massive gravity or the coefficients of the effective action.

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## REFERENCES

[1] Elvang, Henriette, and Yu-tin Huang. Scatering Amplitudes in Gauge Theory and Gravity. Cambridge University Press, 2015.

[2] Witten, Edward. Baryons in the 1/N Expansion. Nuclear Physics B, vol. 160, no. 1, 26 Nov.

1979, pp. 57115., doi:10.1016/0550-3213(79)90232-3.

[3] Caron-Huot, Simon, et al. Strings from Massive Higher Spins: the Asymptotic Uniqueness of the Veneziano Amplitude. Journal of High Energy Physics, vol. 2017, no. 10, 2017, doi:10.1007/jhep10(2017)026.

[4] Arkani-Hamed, Nima, et al. The EFThedron. BHI Conference 2018.

[5] Huang, Yu-tin, et al. The Space of EFT and CFT: behind the Walls of Cyclic Polytopes. Amplitudes 2018.