1 Introduction

General Relativity has been one of the pilars of modern physics for over 100 years now. Testing the theory and its consequences is therefore very important to solidifying our understand and exploring new areas of physics. However, there are other theories of gravitation which one can study and compare to GR. One such set of theories is the class of scalar-tensor theories of gravity. These theories agree with GR in weak field limits, but can deviate from GR in stronger fields. Such a case where we would see a measurable difference between GR and such scalar-tensor theories are in binary pulsars. We'll study the post-Keplerian timing parameters describing a binary pulsar system in both GR and various scalar-tensor theories of gravity in order to put restrictions on which of these theories agree with experimental data.

2 Post-Keplerian Timing Parameters for General Relativity

Before writing down the Post-Keplerian (PK) parameters for general relativity we first list all relevant variables and what they represent.

ω	Longitude of periastron
$\dot{\omega}$	Advance of periastron
P_b	Orbital period
$\dot{P}_b$	Orbital period derivative
γ	Gravitational redshift
e	Eccentricity
r	Range of Shapiro delay
s	Shape of Shapiro delay
i	Angle of Inclination
x	Projected semi-major axis
m_A	Pulsar mass (measured in units of $M_\odot$)
m_B	Companion mass (measured in units of $M_\odot$)
M	Total mass ($M = m_1 + m_2$)

In general relativity, the equations describing the PK parameters according to [1][2] are

$$\begin{aligned}
\dot{\omega} &= 3\left(\frac{P_b}{2\pi}\right)^{-5/3} G^{2/3}M^{2/3}(1-e^2)^{-1}, && (1)\\
\gamma &= e\left(\frac{P_b}{2\pi}\right)^{1/3} G^{2/3}M^{-1/3}m_B\left(1+\frac{m_2}{M}\right), && (2)\\
\dot{P}_b &= -\frac{192\pi}{5}\left(\frac{P_b}{2\pi}\right)^{-5/3} m_A m_B G^{5/3}M^{-1/3}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)(1-e^2)^{-7/2}, && (3)\\
s &= x\left(\frac{P_b}{2\pi}\right)^{-2/3} G^{-1/3}M^{2/3}m_B^{-1}, && (4)\\
r &= Gm_B. && (5)
\end{aligned}$$

where $s = \sin i$, G is Newton's constant and the convention that $\frac{M_\odot}{c^3} = 1$ is employed.

3 Tensor-scalar theories [3]

Let us consider a general tensor-scalar action involving the metric $\tilde{g}^{\mu\nu}$ (with signature 'mostly plus'), a scalar field Φ, and some matter variables ψ_m (including gauge bosons):

$$S = (16\pi G)^{-1} \int d^4x(-\tilde{g})^{1/2}[F(\Phi)\tilde{R} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial\nu\Phi - U(\Phi)] + S_m[\psi_m; \tilde{g}_{\mu\nu}]. \quad (6)$$

For simplicity, we assume here that the weak equivalence principle is satisfied, i.e., that the matter variables ψ_m are all coupled to the same 'physical metric' $\tilde{g}_{\mu\nu}$. The general model (6) involves three arbitrary functions: a function $F(\Phi)$ coupling the scalar Φ to the Ricci scalar of $\tilde{g}_{\mu\nu}$, $\tilde{R} \equiv R(\tilde{g}_{\mu\nu})$, a function $Z(\Phi)$ renormalizing the kinetic term of Φ, and a potential $U(\Phi)$. As we have the freedom of arbitrary redefinitions of the scalar field, $\Phi \to \Phi^{'} = f(\Phi)$, only two functions among F, Z and U are independent. It is ofetn convenient to rewrite (6) in a canonical form, obtained by redefining both Φ and $\tilde{g}_{\mu\nu}$ according to

$$g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu}, \quad (7)$$

$$\varphi = \pm \int d\Phi \left[\frac{3}{4}\frac{F^{'2}(\Phi)}{F^2(\Phi)} + \frac{1}{2}\frac{Z(\Phi)}{F(\Phi)}\right]^{1/2}. \quad (8)$$

This yields

$$S = (16\pi G)^{-1} \int d^4x(-g)^{1/2}[R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)] + S_m[\psi_m; A^2(\varphi)g_{\mu\nu}]. \quad (9)$$

where

$$R \equiv R(g_{\mu\nu}), \quad (10)$$

$$V(\varphi) = F^{-2}(\Phi)U(\Phi), \quad (11)$$

$$A(\varphi) = F^{-1/2}(\Phi) \quad (12)$$

with $\Phi(\varphi)$ obtained by inverting the integral (8).

The two arbitrary functions entering the canonical form (9) are: (i) the conformal coupling function $A(\varphi)$ and (ii) the potential function $V(\varphi)$. Note that the 'physical metric $\tilde{g}_{\mu\nu}$ (the one measured by laboratory clocks and rods) is conformally related to the 'Einstein metric' $g_{\mu\nu}$, being given by $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}$.
In many technical developments it is useful to work with the logarithmic coupling function $a(\varphi)$ such that:

$$a(\varphi) \equiv \ln A(\varphi). \quad (13)$$

In the case of the general model (6) this logarithmic coupling is given by

$$a(\varphi) = -\frac{1}{2}\ln F(\Phi) \quad (14)$$

where $\Phi(\varphi)$ must be obtained from (8).

4 Post-Keplerian Timing Parameters for Brans-Dicke Theory [1][3]

Let's now consider the case of the Jordan-Fierz-Brans-Dicke action, which is of the general type (6) with

$$F(\Phi) = \Phi \quad (15)$$

$$Z(\Phi) = \omega_{BD}\Phi^{-1}, \quad (16)$$

where ω_{BD} is an arbitrary constant. Using eqns. (9) and (12), one finds that -2$\alpha_0\varphi = \ln\Phi$ and that the logarithmic coupling function is simply

$$a(\varphi) = \alpha_0\varphi + const., \quad (17)$$

where $\alpha_0^2 = (2\omega_{BD}+3)^{-1}$.

It's useful to note that the larger the value of ω_{BD}, the smaller the effects of the scalar field, and in the limit $\omega_{BD} \to \infty$, the theory becomes indistinguishable from GR in all its predictions. The equations for the PK parameters in Brans-Dicke theory are written in terms of the variables listed in section 1 as well as two others, as we now define.

$$s_X = -\left(\frac{\partial(\ln m_X)}{\partial(\ln G)}\right)_N, \tag{18}$$

$$\kappa_X = -\left(\frac{\partial(\ln I_X)}{\partial(\ln G)}\right)_N. \tag{19}$$

for $X = \{A, B\}$ The quantities s_X and κ_X measure the "sensitivity" of the masses m_X and moment of inertial I_X of each body to changes in the scalar field (reflected in changes in G) for a fixed baryon number N. The quantity s_A is related to the gravitation binding energy. The sensitivities will depend on the neutron-star equation of state. We are now ready to write down the equations of the PK parameters in Brans-Dicke theory, but before doing so, let's make some definitions in terms of our variables to simplify the equations.

$$\xi = (2+\omega_{BD})^{-1} \tag{20}$$

$$\mathcal{G} = 1-\xi(s_A+s_B-2s_As_B) \tag{21}$$

$$\mathcal{P} = \mathcal{G}\left[1-\frac{2}{3}\xi+\frac{1}{3}\xi(s_A+s_B-2s_As_B)\right] \tag{22}$$

$$\rho = 1-\xi s_B \tag{23}$$

$$\eta = (1-2s_B)\xi \tag{24}$$

$$\Gamma = 1-2(m_As_B+m_Bs_A)/M \tag{25}$$

$$\Gamma' = 1-s_A-s_B \tag{26}$$

$$k_1 = \mathcal{G}^2\left[12\left(1-\frac{1}{2}\xi\right)+\xi\Gamma^2\right] \tag{27}$$

$$k_2 = \mathcal{G}^2\left[11\left(1-\frac{1}{2}\xi\right)+\frac{1}{2}\xi\left(\Gamma^2-5\Gamma\Gamma'-\frac{15}{2}\Gamma'^2\right)\right] \tag{28}$$

$$F(e) = \frac{1}{12}(1-e^2)^{7/2}\left[k_1\left(1+\frac{7}{2}e^2+\frac{1}{2}e^4\right)-k_2\left(\frac{1}{2}e^2+\frac{1}{8}e^4\right)\right] \tag{29}$$

$$G(e) = (1-e^2)^{-5/2}\left(1+\frac{1}{2}e^2\right) \tag{30}$$

$$\zeta = s_A-s_B \tag{31}$$

With these definitions in place, we are now ready to write down the equations for the PK parameters in Brans-Dicke theory[1].

$$\dot{\omega} = 3\left(\frac{P_b}{2\pi}\right)^{-5/3} G^{2/3}M^{2/3}(1-e^2)^{-1}\mathcal{P}\mathcal{G}^{-4/3} \tag{32}$$

$$\gamma = e\left(\frac{P_b}{2\pi}\right)^{1/3} m_BG^{2/3}(M\mathcal{G})^{-1/3}(\rho+\mathcal{G}m_BM^{-1}+\kappa_A\eta) \tag{33}$$

$$\dot{P}_b = -\frac{192\pi}{5}\left(\frac{P_b}{2\pi}\right)^{-5/3} m_Am_BG^{5/3}M^{-1/3}\mathcal{G}^{-4/3}F(e)-4\pi\left(\frac{P_b}{2\pi}\right)^{-1} m_Am_BM^{-2}\xi\zeta^2G(e) \tag{34}$$

We should check that these equations reduce down to equations (1)-(3) in the limit that Brans-Dicke theory becomes General Relativity. So as $\omega_{BD} \to \infty$, we see that

$$
\begin{aligned}
\xi &\rightarrow 0, && (35)\\
\mathcal{G} &\rightarrow 1, && (36)\\
\mathcal{P} &\rightarrow 1, && (37)\\
\rho &\rightarrow 1, && (38)\\
\eta &\rightarrow 0, && (39)\\
k_1 &\rightarrow 12, && (40)\\
k_2 &\rightarrow 11, && (41)\\
F(e) &\rightarrow (1-e^2)^{-7/2}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right), && (42)\\
& && (43)
\end{aligned}
$$

and plugging these into equations (32)-(34) we see they do indeed reduce to equations (1)-(3).

5 Post-Keplerian Timing Parameters for Tensor-Scalar Theory of Gravity with $A(\varphi) = \exp(\alpha_0\varphi + \frac{1}{2}\beta_0\varphi^2)$[3]

Before writing down the desired equations, we first make a couple notes and definitions

If one uses appropriate units in the asymptotic region far from the system, namely units such that the asymptotic value $a(\varphi_0)$ of $a(\varphi)$ vanishes, all observable quantities at the post-Newtonian (1PN) level depend only on the values of the first two derivatives of $a(\varphi)$ at $\varphi = \varphi_0$. More precisely, we define

$$\alpha(\varphi) \equiv \frac{\partial a(\varphi)}{\partial\varphi}; \beta(\varphi) \equiv \frac{\partial\alpha(\varphi)}{\partial\varphi} = \frac{\partial^2 a(\varphi)}{\partial\varphi^2} \tag{44}$$

and denote $\alpha_0 \equiv \alpha(\varphi_0), \beta_0 \equiv \beta(\varphi_0)$

The field equations of a general tensor-scalar theory, as derived from the canonical action (9) (neglecting the effect of $V(\varphi)$) read

$$
\begin{aligned}
R_{\mu\nu} &= 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi G\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) && (45)\\
\Box_g\varphi &= -4\pi G\alpha(\varphi)T && (46)
\end{aligned}
$$

where $T^{\mu\nu} \equiv 2(-g)^{-1/2}\delta S_m/\delta g_{\mu\nu}$ denotes the material stress-energy tensor in 'Einstein units'. All tensorial operations in eqns. (45) and (46) are performed by using the Einstein metric $g_{\mu\nu}$.

Explicitly writing the field equations (45) and (46) for a slowly rotating (stationary, axisymmetric) neutron star, labeled A, leads to a coupled set of ordinary differential equations constraining the radial dependence of $g_{\mu\nu}$ and φ. Imposing the boundary conditions $g_{\mu\nu} \rightarrow \eta_{\mu\nu}, \varphi \rightarrow \varphi_a$ at large radial distances, finally determines the crucial 'form factors' (in Einstein units) describing the effective coupling between the neutron star A and the fields to which it is sensitive: total mass $m_A(\varphi_a)$, total scalar charge $\omega_A(\varphi_a)$, and moment of inertia $I_A(\varphi)$. This φ_a is a combination of the cosmological background value φ_0 and of the scalar influence of the companion of the considered neutron star. As indicated, these quantities are functions of the asymptotic value φ_a of φ felt by the considered neutron star. They satisfy the relation $\omega_A = -\partial m_A(\varphi_a)/\partial\varphi_a$. From them, one defines other quantities that play an important role in binary pulsar physics, notably

$$\alpha_A(\varphi_a) \equiv -\frac{\omega_A}{m_A} \equiv \frac{\partial \ln m_A}{\partial \varphi_a} \tag{47}$$

$$\beta_A(\varphi_a) \equiv \frac{\partial \alpha_A}{\partial \alpha_a} \tag{48}$$

$$\kappa_A(\varphi_a) \equiv -\frac{\partial \ln I_A}{\partial \varphi_a} \tag{49}$$

The quantity α_A plays a crucial role. It measures the effective coupling strength between the neutron star and the ambient scalar field. If we formally let the self-gravity of the neutron A tend toward zero (i.e., if we consider a weakly self-gravitating object), the function $\alpha_A(\varphi_a)$ becomes replaced by $\alpha(\varphi_a)$ where $\alpha(\varphi) \equiv \partial a(\varphi)/\partial\varphi$ is the coupling strength. Roughly speaking, we can think of $\alpha_A(\varphi_a)$ as a (suitable defined) average value of the local coupling strength $\alpha(\varphi(r))$ over the radial profile of the neutron star A.

For our purposes we will indeed make the assumption that $\alpha_A(\varphi_a) = \alpha_B(\varphi_a) = \alpha(\varphi_a) = \alpha_0$. This is equivalent to the statement that $\partial \ln \widetilde{m}_A(\varphi_a)/\partial\varphi_a = 0$, where $\widetilde{m}_A$ is the mass in the Jordan frame, since $m_A(\varphi_a) = A(\varphi_a)\widetilde{m}_A(\varphi_a)$, and similarly for m_B. Following the same line of reasoning, we'll take $\beta_A(\varphi_a) = \beta_B(\varphi_a) = \beta(\varphi_a) = \beta_0$

Noting one more time that the label A refers to the object which is being timed (the pulsar), the label B refers to its companion and x (see table in section 1) denotes the projected semi-major axis of the orbit of A, we are now ready to write down the equations of the PK parameters.

$$\dot{\omega} = \frac{3}{1-e^2}\left(\frac{P_b}{2\pi}\right)^{-5/3}(GM)^{2/3}\left(\frac{6+(4-\beta_0)\alpha_0^2-2\alpha_0^4}{6(1+\alpha_0^2)^{4/3}}\right) \tag{50}$$

$$\gamma = e\left(\frac{P_b}{2\pi}\right)^{1/3} G^{2/3}M^{-1/3}m_B(1+\alpha_0^2)^{-1/3}\left(1+\frac{m_B}{M}(1+\alpha_0^2)+\kappa_A\alpha_0\right) \tag{51}$$

$$\begin{aligned}\dot{P}_b = & -\frac{\pi m_A m_B \alpha_0^2 G^{5/3}}{3M^{1/3}(1+\alpha_0^2)^{4/3}}\left(\frac{P_b}{2\pi}\right)^{-5/3}\frac{e^2(1+e^2/4)(8(1+\alpha_0^2)-6\beta_0)^2}{(1-e^2)^{7/2}} \\ & - \frac{32\pi m_A m_B G^{5/3}}{5M^{1/3}}\left(\frac{P_b}{2\pi}\right)^{-5/3}\frac{(1+73e^2/24+37e^4/96)(1+\alpha_0^2)^{2/3}}{(1-e^2)^{7/2}}(6+\alpha_0^2)\end{aligned} \tag{52}$$

$$s = x\left(\frac{P_b}{2\pi}\right)^{-2/3}(G(1+\alpha_0^2))^{-1/3}M^{-4/3}m_B^{-1} \tag{53}$$

$$r = G(1+\alpha_0^2)m_B \tag{54}$$

References

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